RÉNYI DIMENSION AND GAUSSIAN FILTERING

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ABSTRACT. Consider the partition function $S_\mu^q(\epsilon)$ associated in theory of Rényi dimension to a finite Borel measure μ on Euclidean d-space. This partion function $S_\mu^q(\epsilon)$ is the sum of the q-th powers of the measure applied to a partition of d-space into d-cubes of width ϵ. We further Guerin's investigation of the relation between this partition function and the Lebesgue L^p norm (L^q norm) of the convolution of μ against an approximate identity of Gaussians. We prove a Lipschitz-type esimate on the partition function. This bound on the partition function leads to results regarding the computation of Rényi dimension. It also shows that the partion function is of O-regular variation.

We find situtations where one can or cannot replace the partition function by a discrete version. We discover that the slopes of the least-square best fit linear approximations to the partion function cannot always be used to calculate upper and lower Rényi dimension.

(preprint version)

1. Introduction

The Rényi dimensions of a finite Borel measure μ on $\mathbb{R}^d$ are derived from slopes of certain long secants of the log-log plot of the function

$$\epsilon \mapsto S_\mu^q(\epsilon) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \mu(\epsilon\mathbf{k} + \epsilon\mathbb{I})^q,$$

where

$$\mathbb{I} = [0,1) \times [0,1) \times \cdots \times [0,1).$$

There are exceptions for $q = 0, 1$. In this paper we only address the cases $0 < q < 1$ and $1 < q < \infty$. In this introduction we wish to avoid convergence issues, so let us also assume that μ has bounded support.

For any x_0, we set

$$D_q^\pm(\mu) = \lim_{x\to-\infty} {}^{\sup}_{\inf} \frac{1}{q-1} \frac{\ln\left(S_\mu^q(e^x)\right) - \ln\left(S_\mu^q(e^{x_0})\right)}{x - x_0}.$$

The constant terms are irrelevant, so this is usually written as

$$D_q^\pm(\mu) = \lim_{x\to-\infty} {}^{\sup}_{\inf} \frac{1}{q-1} \frac{\ln\left(S_\mu^q(e^x)\right)}{x}$$

or

$$D_q^\pm(\mu) = \lim_{\epsilon\to0} {}^{\sup}_{\inf} \frac{1}{q-1} \frac{\ln\left(\sum_{\mathbf{k}\in\mathbb{Z}^d} \mu(\epsilon\mathbf{k} + \epsilon\mathbb{I})^q\right)}{\ln(\epsilon)}.$$

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Moving an exponent inside the log gives

$$D_q^{\pm}(\mu) = \lim_{\epsilon\to 0} {}^{\sup}_{\inf} \frac{q}{q-1} \frac{\ln\left(\left\| \mathbf{k} \mapsto \mu(\epsilon\mathbf{k} + \epsilon\mathbb{I}) \right\|_q \right)}{\ln(\epsilon)}.$$

This shows a relationship between convolution, L^p-norms and Rényi dimension, because

$$(\chi|_{(-\epsilon\mathbb{I})} * \mu)(\mathbf{x}) = \mu(\mathbf{x} + \epsilon\mathbb{I}).$$

Here we have used $\chi|_{(-\epsilon\mathbb{I})}$ to denote the characteristic function of $-\epsilon\mathbb{I}$.

Guérin ([6]) showed a more general relation between convolutions, L^p-norms and Rényi dimensions. He showed that for many choices of a scalar-valued function g on $\mathbb{R}^d$, if

$$1 < q < \infty,$$

and if we set

$$g_\epsilon(\mathrm{x}) = \epsilon^{-d} g(\epsilon^{-1}\mathbf{x}),$$

then

$$D_q^{\pm}(\mu) = \lim_{\epsilon\to 0} {}^{\sup}_{\inf} \frac{1}{q-1} \frac{\ln\left(\epsilon^{d(q-1)} \left\| g_\epsilon * \mu \right\|_q^q \right)}{\ln(\epsilon)},$$

or

$$D_q^{\pm}(\mu) = d + \lim_{\epsilon\to 0} {}^{\sup}_{\inf} \frac{q}{q-1} \frac{\ln\left(\left\| g_\epsilon * \mu \right\|_q \right)}{\ln(\epsilon)}. \tag{1}$$

Guérin allowed g from a large class of complex-valued, rapidly decreasing functions.

A technical improvement on Guérin's result is given in Section 2, with additional restrictions on g but allowing $0 < q < 1$. For a given μ, and a "nice" function $g \geq 0$, we establish a uniform bound on difference

$$\ln\left(\epsilon^{d(q-1)} \left\| g_\epsilon * \mu \right\|_q^q \right) - \ln\left(S_\mu^q(\epsilon) \right).$$

This estimate allows us to analyze sequences

$$\frac{\ln\left(S_\mu^q(\epsilon_n) \right)}{\ln(\epsilon_n)}$$

by looking instead at

$$\frac{\ln\left(\left\| g_{\epsilon_n} * \mu \right\|_q \right)}{\ln(\epsilon_n)}.$$

This will be advantageous if we choose g_ϵ properly.

Most importantly, we wish to let g be a standard Gaussian on $\mathbb{R}^d$. As we have the convention

$$g_\epsilon(\mathrm{x}) = \epsilon^{-d} g(\epsilon^{-1}\mathbf{x}),$$

the semigroup rule ends up as

$$g_\epsilon * g_\eta = g_{\sqrt{\epsilon^2+\eta^2}}.$$

We find that $\left\| g_\epsilon * \mu \right\|_q$ gives us information not apparent in the sums $S_\mu^q(\epsilon)$. Specifically, we find constants A and B so that

$$\left| \ln\left(S_\mu^q(e^x) \right) - \ln\left(S_\mu^q(e^y) \right) \right| \leq A + B|x - y|.$$

It follows that $S_\mu^q(x^{-1})$ is of O-regular variation.

Generalizing a result of Riedi ([15]), we show that

$$\frac{1}{q-1}\frac{\ln\left(S_{\mu}^{q}(\epsilon_n)\right)}{\ln(\epsilon_n)}$$

can be used to calculate $D_q^{\pm}(\mu)$, even if ϵ_n converges to zero somewhat faster than geometrically. The specific requirement is that

$$\lim_{n\to\infty}\frac{\ln(\epsilon_{n+1})}{\ln(\epsilon_n)} > 1.$$

Examples are exhibited that shows that this result is in some sense the best possible.

We also give some new estimates on the Rényi dimensions of a convolution $\mu * \nu$ in terms of the Rényi dimensions of μ and ν.

In Sections 7 and 8 we consider some of the changes that occur when one replaces

$$\frac{\ln\left(S_{\mu}^{q}(e^{x})\right)-\ln\left(S_{\mu}^{q}(e^{x_0})\right)}{x-x_0}$$

by the slope of a least-squares best fit line over $[x, x_0]$ to the function

$$t \mapsto \ln\left(S_{\mu}^{q}(e^{t})\right).$$

We exhibit an example where these least-squares slopes do not determine the upper Rényi dimension.

The examples we give have features that occur only on scales that grow doubly exponentially. In the final section we suggest an alteration of Rényi that better detects the aberrant nature of these examples.

2. Norms after convolution

Here follows our main technical result. Our initial interest here was in the context of image analysis, were convolution by scaled Gaussians is common. For example, see [3].

For $1 < q < \infty$, we have an easy finite bound on the partition function

$$S_{\mu}^{q}(\epsilon) = \sum_{\mathbf{k}\in\mathbb{Z}^d}\mu(\epsilon\mathbf{k}+\epsilon\mathbb{I})^q,$$

specifically

$$\begin{aligned} S_{\mu}^{q}(\epsilon) &\leq \left(\sum_{\mathbf{k}\in\mathbb{Z}^d}\mu(\epsilon\mathbf{k}+\epsilon\mathbb{I})\right)^q \\ &= \mu(\mathbb{R}^d)^q \\ &< \infty. \end{aligned}$$

For $0 < q < 1$, it is possible to have $S_{\mu}^{q}(\epsilon) = \infty$.

Definition 2.1. A finite Borel measure μ on $\mathbb{R}^d$ is *q-finite* if $S_{\mu}^{q}(\epsilon) < \infty$. Notice that if μ has bounded support then μ is q-finite for all $0 < q < 1$.

Barbaroux, Germinet, and Tcheremchantsev have the following result implicitly in [1].

Lemma 2.2. *Let μ be a finite Borel measure on $\mathbb{R}^d$. For any $0 < q < 1$, the following are equivalent:*

(a) *μ is q-finite;*

(b) *there exists* $\epsilon > 0$ *for which* $S_\mu^q(\epsilon) < \infty$;
(c) *for all* $\epsilon > 0$ *it is true that* $S_\mu^q(\epsilon) < \infty$.

Proof. On page 992-3 of [1] it is shown that if μ is q-finite then $S_\mu^q(\epsilon) < \infty$ for small ϵ. Also, it is shown that if μ is not q-finite then $S_\mu^q(\epsilon) = \infty$ for small ϵ. A rescaling argument show that if $S_\mu^q(\epsilon)$ is ever finite then it is finite for all small ϵ, while if it is ever infinite, it is infinite for all small ϵ. Therefore, the partition function is either finite for all ϵ or infinite for all ϵ. □

The proof of the following borrows from the methods in [1].

Lemma 2.3. *Suppose g is a real-valued Borel measurable function on $\mathbb{R}^d$ that is nonnegative, bounded, bounded away from zero in a neighborhood of $\mathbf{0}$, and rapidly decreasing. Let*

$$g_\epsilon(\mathbf{x}) = \epsilon^{-d} g(\epsilon^{-1}\mathbf{x}).$$

Suppose μ is a finite Borel measure on $\mathbb{R}^d$. If $1 < q < \infty$, or if $0 < q < 1$ and μ is q-finite, then there exists a constant $1 < C < \infty$ so that for all positive ϵ,

$$C^{-1} \leq \frac{\epsilon^{d(q-1)} \left\| g_\epsilon * \mu \right\|_q^q}{S_\mu^q(\epsilon)} \leq C. \tag{2}$$

Here the L^q norm is with respect to Lebesgue measure. Therefore

$$D_q^{\pm}(\mu) = d + \lim_{\epsilon \to 0} {}^{sup}_{inf} \frac{q}{q-1} \frac{\ln\left(\left\| g_\epsilon * \mu \right\|_q\right)}{\ln(\epsilon)}.$$

Proof. We will use m to denote Lebesgue measure, to keep it straight from μ. Let us denote the open unit rectangle at the origin by $\mathbb{D}$, so

$$\mathbb{D} = (-1,1) \times (-1,1) \times \cdots \times (-1,1).$$

Recall $\mathbb{I}$ is the product of d copies of $[0,1)$.

Let us denote by $\mu^{(\epsilon)}$ the sequences over $\mathbb{Z}^d$ given by

$$\mu_{\mathbf{n}}^{(\epsilon)} = \mu(\epsilon\mathbf{n} + \epsilon\mathbb{I}).$$

Thus

$$S_\mu^q(\epsilon) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \mu(\epsilon\mathbf{k} + \epsilon\mathbb{I})^q = \left\| \mu^{(\epsilon)} \right\|_q^q$$

(the norm here is on $l^q(\mathbb{Z}^d)$).

An obvious rescaling reduces this theorem to the special case where

$$\inf\{g(\mathbf{x}) \mid \mathbf{x} \in \mathbb{D}\} > 0,$$

so let us make this assumption. We compute

$$\left\| g_\epsilon * \mu \right\|_q^q = \epsilon^{-qd} \sum_{\mathbf{j} \in \mathbb{Z}^d} \int_{\epsilon\mathbf{j} + \epsilon\mathbb{I}} \left(\sum_{\mathbf{k} \in \mathbb{Z}^d} \int_{\epsilon\mathbf{k} + \epsilon\mathbb{I}} g(\epsilon^{-1}(\mathbf{x} - \mathbf{y}))\, d\mu(\mathbf{y}) \right)^q dm(\mathbf{x}). \tag{3}$$

If

$$\mathbf{x} \in \epsilon\mathbf{j} + \epsilon\mathbb{I}$$

and

$$\mathbf{y} \in \epsilon\mathbf{k} + \epsilon\mathbb{I}$$

then

$$\epsilon^{-1}(\mathbf{x} - \mathbf{y}) \in (\mathbf{j} - \mathbf{k}) + \mathbb{D}.$$

Let us define γ and Γ as sequences over $\mathbb{Z}^d$, by

$$\gamma_{\mathbf{n}} = \inf\{g(\mathbf{x}) \mid \mathbf{x} \in \mathbf{n} + \mathbb{D}\}$$

and

$$\Gamma_{\mathbf{n}} = \sup\{g(\mathbf{x}) \mid \mathbf{x} \in \mathbf{n} + \mathbb{D}\}.$$

These give us bounds on the $g(\epsilon^{-1}(\mathbf{x} - \mathbf{y}))$ term inside integrals in (3).

For an upper bound, we get

$$\begin{aligned} \|g_\epsilon * \mu\|_q^q &\leq \epsilon^{-qd} \sum_{\mathbf{j}\in\mathbb{Z}^d} \left(\sum_{\mathbf{k}\in\mathbb{Z}^d} \Gamma_{\mathbf{j}-\mathbf{k}} \mu(\epsilon\mathbf{k} + \epsilon\mathbb{I}) \right)^q \epsilon^d \\ &= \epsilon^{d(1-q)} \left\| \Gamma * \mu^{(\epsilon)} \right\|_q^q . \end{aligned}$$

For a lower bound,

$$\|g_\epsilon * \mu\|_q^q \geq \epsilon^{d(1-q)} \left\| \gamma * \mu^{(\epsilon)} \right\|_q^q .$$

First assume $1 < q < \infty$. The assumption that g is nonzero on $\mathbb{D}$ is here used to obtain $\|\gamma\|_q > 0$. From the rapidly decreasing assumption we obtain $\|\Gamma\|_1 < \infty$. Since,

$$\begin{aligned} \|g_\epsilon * \mu\|_q &\leq \epsilon^{d\frac{(1-q)}{q}} \left\| \Gamma * \mu^{(\epsilon)} \right\|_q \\ &\leq \epsilon^{d\frac{(1-q)}{q}} \|\Gamma\|_1 \left\| \mu^{(\epsilon)} \right\|_q \end{aligned}$$

and

$$\begin{aligned} \|g_\epsilon * \mu\|_q &\geq \epsilon^{d\frac{(1-q)}{q}} \left\| \gamma * \mu^{(\epsilon)} \right\|_q \\ &\geq \epsilon^{d\frac{(1-q)}{q}} \|\gamma\|_q \left\| \mu^{(\epsilon)} \right\|_q , \end{aligned}$$

we may take

$$C = \max\left(\|\Gamma\|_1^q , \|\gamma\|_q^{-q} \right).$$

Now assume $0 < q < 1$. The assumptions on g tell us $\|\gamma\|_1 > 0$ and $\|\Gamma\|_q < \infty$. In this case we may take

$$C = \max\left(\|\Gamma\|_q^q , \|\gamma\|_1^{-q} \right).$$

□

3. Bounding the partition function

In this section we use Lemma 2.3 to establish bounds on

$$\ln\left(S_\mu^q(e^{x+h})\right) - \ln\left(S_\mu^q(e^x)\right)$$

that are of first order in h and hold for all x. Recall that the partition function for μ is

$$S_\mu^q(\epsilon) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \mu(\epsilon\mathbf{k} + \epsilon\mathbb{I})^q.$$

Much of this section is familiar, but one conclusion seems novel: $S_\mu^q(x^{-1})$ is almost decreasing. By this we mean

$$\sup_{x<y} \frac{S_\mu^q(y^{-1})}{S_\mu^q(x^{-1})} < \infty.$$

(See [2].)

In this section, g denotes a standard Gaussian, and for $\epsilon > 0$ we set

$$g_\epsilon(\mathbf{x}) = \epsilon^{-d} g(\epsilon^{-1}\mathbf{x}).$$

Let us also adopt the notation

$$I_\mu^q(\epsilon) = \|g_\epsilon * \mu\|_q^q .$$

Lemma 3.1. *Suppose μ is a finite Borel measure. If $1 < q < \infty$ then*

$$\epsilon_1 \leq \epsilon_2 \implies \|g_{\epsilon_2} * \mu\|_q \leq \|g_{\epsilon_1} * \mu\|_q .$$

If $0 < q < 1$, and μ is q-finite, then

$$\epsilon_1 \leq \epsilon_2 \implies \|g_{\epsilon_2} * \mu\|_q \geq \|g_{\epsilon_1} * \mu\|_q .$$

Proof. If $1 < q < \infty$ then

$$\begin{aligned} \|g_\eta * \mu\|_q &= \left\| g_{\sqrt{\eta^2-\epsilon^2}} * g_\epsilon * \mu \right\|_q \\ &\leq \left\| g_{\sqrt{\eta^2-\epsilon^2}} \right\|_1 \|g_\epsilon * \mu\|_q \\ &= \|g_\epsilon * \mu\|_q . \end{aligned}$$

For $0 < q < 1$ the middle step becomes

$$\left\| g_{\sqrt{\eta^2-\epsilon^2}} * g_\epsilon * \mu \right\|_q \geq \left\| g_{\sqrt{\eta^2-\epsilon^2}} \right\|_1 \|g_\epsilon * \mu\|_q .$$

□

Lemma 3.2. *Suppose μ is a finite Borel measure and that n is a natural number. For $1 < q < \infty$, if $\epsilon > 0$ then*

$$0 \leq \ln\left(S_\mu^q(\epsilon)\right) - \ln\left(S_\mu^q(2^{-n}\epsilon)\right) \leq n \ln\left(2^{d(q-1)}\right).$$

For $0 < q < 1$, if μ is q-finite and $\epsilon > 0$ then

$$n \ln\left(2^{d(q-1)}\right) \leq \ln\left(S_\mu^q(\epsilon)\right) - \ln\left(S_\mu^q(2^{-n}\epsilon)\right) \leq 0.$$

Proof. Suppose $1 < q < \infty$. Given a disjoint union of Borel sets,

$$E_1 \cup \cdots \cup E_{2^d} = F$$

we have the estimates

$$2^{d(1-q)} \mu(F)^q \leq \sum_{j=1}^{2^d} \mu(E_j)^q \leq \mu(F)^q.$$

Therefore

$$\ln\left(S_\mu^q(\epsilon)\right) + \ln(2^{d(1-q)}) \leq \ln\left(S_\mu^q(2^{-1}\epsilon)\right) \leq \ln\left(S_\mu^q(\epsilon)\right).$$

For $0 < q < 1$ the inequalities are all easily reversed. □

Lemma 3.3. *Suppose μ is a finite Borel measure. If $1 < q < \infty$ then there is a constant E so that*

$$|\ln(\epsilon) - \ln(\eta)| \leq \ln(2)$$

implies

$$\left|\ln\left(S_\mu^q(\epsilon)\right) - \ln\left(S_\mu^q(\eta)\right)\right| \leq E.$$

If $0 < q < 1$, and μ is q-finite, then there is constant E so that

$$|\ln(\epsilon) - \ln(\eta)| \leq \ln(2)$$

implies

$$\left|\ln\left(S_\mu^q(\epsilon)\right) - \ln\left(S_\mu^q(\eta)\right)\right| \leq E.$$

Proof. We may assume

$$2^{-1}\epsilon \leq \eta \leq \epsilon.$$

By Theorem 2.3, there is a constant C so that for all $\rho > 0$,

$$\left|d(q-1)\ln(\rho) + \ln\left(I_\mu^q(\rho)\right) - \ln\left(S_\mu^q(\rho)\right)\right| \leq D.$$

If we set

$$Q = d(q-1)\ln\left(2^{-\frac{1}{2}}\epsilon\right)$$

and

$$D = d|q-1|\ln\left(2^{\frac{1}{2}}\right)$$

then

$$2^{-1}\epsilon \leq \rho \leq \epsilon \implies \left|Q + \ln\left(I_\mu^q(\rho)\right) - \ln\left(S_\mu^q(\rho)\right)\right| \leq D.$$

Now assume $1 < q < \infty$. Lemma 3.1 gives us

$$\ln\left(I_\mu^q(\epsilon)\right) \leq \ln\left(I_\mu^q(\eta)\right) \leq \ln\left(I_\mu^q(2^{-1}\epsilon)\right)$$

and Lemma 3.2 gives us

$$\ln\left(S_\mu^q(2^{-1}\epsilon)\right) \leq \ln\left(S_\mu^q(\epsilon)\right).$$

We put this information together as

$$\begin{aligned}
\ln\left(S_\mu^q(\epsilon)\right) - 2D_1 &\leq Q + \ln\left(I_\mu^q(\epsilon)\right) - D \\
&\leq Q + \ln\left(I_\mu^q(\eta)\right) - D \\
&\leq \ln\left(S_\mu^q(\eta)\right) \\
&\leq Q + \ln\left(I_\mu^q(\eta)\right) + D \\
&\leq Q + \ln\left(I_\mu^q(2^{-1}\epsilon)\right) + D \\
&\leq \ln\left(S_\mu^q(2^{-1}\epsilon)\right) + 2D \\
&\leq \ln\left(S_\mu^q(\epsilon)\right) + 2D
\end{aligned}$$

For $0 < q < 1$ the proof is similar. □

Theorem 3.4. *Suppose μ is a finite Borel measure on $\mathbb{R}^d$. Let $B = d(q-1)$. For $1 < q < \infty$, there is constant A so that*

$$0 < \eta < \epsilon$$

implies

$$-A \leq \ln\left(S_\mu^q(\epsilon)\right) - \ln\left(S_\mu^q(\eta)\right) \leq A + B(\ln(\epsilon) - \ln(\eta)).$$

For $0 < q < 1$, if μ is q-finite then there is constant A so that

$$0 < \eta < \epsilon$$

implies

$$-A + B(\ln(\epsilon) - \ln(\eta)) \leq \ln\left(S_\mu^q(\epsilon)\right) - \ln\left(S_\mu^q(\eta)\right) \leq A.$$

Proof. Assume first that $1 < q < \infty$. For some natural number n,

$$2^{-n-1}\epsilon \leq \eta \leq 2^{-n}\epsilon.$$

This means

$$n \leq \frac{1}{\ln(2)}(\ln(\epsilon) - \ln(\eta)) + 1$$

and so by the last three lemmas,

$$\begin{aligned}
& \ln\left(S_\mu^q(\epsilon)\right) - B(\ln(\epsilon) - \ln(\eta)) - B\ln(2) - E \\
\leq\ & \ln\left(S_\mu^q(\epsilon)\right) - nB\ln(2) - E \\
\leq\ & \ln\left(S_\mu^q(2^{-n}\epsilon)\right) - E \\
\leq\ & S_\mu^q(\eta) \\
\leq\ & \ln\left(S_\mu^q(2^{-n}\epsilon)\right) + E \\
\leq\ & \ln\left(S_\mu^q(\epsilon)\right) + E \\
\leq\ & \ln\left(S_\mu^q(\epsilon)\right) + B\ln(2) + E
\end{aligned}$$

so we can set

$$A = B\ln(2) + E.$$

For $0 < q < 1$ the proof is similar. $\square$

4. Application to discrete limits

Riedi [14, 15] shows that

$$D_p^{\pm}(\mu) = \lim_{n\to\infty} \begin{smallmatrix}\sup\\ \inf\end{smallmatrix} \frac{1}{q-1} \frac{\ln\left(S_\mu^q(\epsilon_n)\right)}{\ln(\epsilon_n)} \tag{4}$$

for $\epsilon_n \searrow 0$, so long as

$$\lim_{n\to\infty}\sup(\ln(\epsilon_n) - \ln(\epsilon_{n+1})) < \infty.$$

Indeed, he works with all $p \in \mathbb{R}$, and shows that grids other than

$$\{\epsilon\mathbf{k} + \epsilon\mathbb{I} \mid \mathbf{k} \in \mathbb{Z}^d\}$$

can be used. What concerns us here is that Riedi showed that (4) is valid for a geometric series. We can go further, to allow sequences such a $\epsilon_n = b^{-n^r}$.

Lemma 4.1. *Suppose μ is a finite Borel measure on $\mathbb{R}^d$, and $q \neq 1$ is a positive number. If $0 < q < 1$ then also suppose $\sum \mu(\mathbf{k}+\mathbb{I})^q$ is finite. If $\epsilon_n \searrow 0$ and $\eta_n \searrow 0$ with*

$$\lim_{n\to\infty} \frac{\ln(\eta_n)}{\ln(\epsilon_n)} = 1,$$

then

$$\lim_{n\to\infty} \left| \frac{\ln\left(S_\mu^q(\eta_n)\right)}{\ln(\eta_n)} - \frac{\ln\left(S_\mu^q(\epsilon_n)\right)}{\ln(\epsilon_n)} \right| = 0$$

Proof. Let A and B be the constants from Theorem 3.4, and let

$$A_0 = A + \left|\ln(S_\mu^p(1)\right|$$

so that

$$\left|\ln\left(S_\mu^q(\epsilon)\right)\right| \leq A_0 + C|\ln(\epsilon)|$$

for all ϵ. We may assume $\epsilon_n \leq e^{-1}$ and $\eta_n \leq e^{-1}$, in which case

$$\begin{aligned}
&\left|\frac{\ln\left(S_\mu^q(\eta_n)\right)}{\ln(\eta_n)} - \frac{\ln\left(S_\mu^q(\epsilon_n)\right)}{\ln(\epsilon_n)}\right| \\
\leq\ & \frac{\left|\ln\left(S_\mu^q(\eta_n)\right) - \ln\left(S_\mu^q(\epsilon_n)\right)\right|}{-\ln(\eta_n)} + \frac{\left|\ln\left(S_\mu^q(\epsilon_n)\right)\right|}{-\ln(\epsilon_n)}\left|\frac{\ln(\epsilon_n)}{\ln(\eta_n)} - 1\right| \\
\leq\ & \frac{A + C|\ln(\eta_n) - \ln(\epsilon_n)|}{-\ln(\eta_n)} + \frac{A_0 - C\ln(\epsilon_n)}{-\ln(\epsilon_n)}\left|\frac{\ln(\epsilon_n)}{\ln(\eta_n)} - 1\right| \\
\leq\ & \frac{-A}{\ln(\eta_n)} + C\left|1 - \frac{\ln(\epsilon_n)}{\ln(\eta_n)}\right| + (A_0 + C)\left|\frac{\ln(\epsilon_n)}{\ln(\eta_n)} - 1\right| \\
\to\ & 0.
\end{aligned}$$

□

Theorem 4.2. *Suppose μ is a finite Borel measure on $\mathbb{R}^d$ and $q \neq 1$ is a positive number. If $0 < q < 1$ then also suppose μ is q-finite. If $\epsilon_n \searrow 0$ and*

$$\lim_{n\to\infty} \frac{\ln(\epsilon_{n+1})}{\ln(\epsilon_n)} = 1,$$

then

$$D_q^\pm(\mu) = \frac{1}{q-1} \lim_{n\to\infty} {}^{sup}_{inf} \frac{\ln\left(S_\mu^q(\epsilon_n)\right)}{\ln(\epsilon_n)}.$$

Moreover, the sequence

$$n \mapsto \frac{\ln\left(S_\mu^q(\epsilon_n)\right)}{\ln(\epsilon_n)} \tag{5}$$

and the net

$$\epsilon \mapsto \frac{\ln\left(S_\mu^q(\epsilon)\right)}{\ln(\epsilon)} \tag{6}$$

have the same accumulation points in $[-\infty, \infty]$.

Proof. Suppose x is an accumulation point of the net in (6). This means there is a sequence $\eta_n \searrow 0$ so that

$$\lim_{n\to\infty} \frac{\ln\left(S_\mu^q(\eta_n)\right)}{\ln(\eta_n)} = x.$$

Let the sequence k_n be defined so that

$$\epsilon_{k_n+1} \leq \eta_n \leq \epsilon_{k_n}.$$

Of course, $k_n \searrow 0$ and

$$\lim_{n\to\infty} \frac{\ln(\eta_n)}{\ln(\epsilon_{k_n})} = 1.$$

The last lemma tells us

$$\lim_{n\to\infty} \frac{\ln\left(S_\mu^q(\epsilon_{k_n})\right)}{\ln(\epsilon_{k_n})} = \lim_{n\to\infty} \frac{\ln\left(S_\mu^q(\eta_n)\right)}{\ln(\eta_n)} = x.$$

Thus x is also an accumulation point of the sequence in (5). $\square$

5. Examples

It is possible to use simple recursive definitions to create a Borel measure μ, with support in $[0,1]$, so that the partition function

$$S_\mu^q(\epsilon) = \sum_{k\in\mathbb{Z}} \mu(\epsilon k + \epsilon\mathbb{I})^q$$

behaves almost any way we would like. However, we do need to respect Lemma 3.2.

Lemma 5.1. *Suppose*

$$0 \le a_n \le 1$$

for $n \ge 1$. *If* $q \ne 1$ *is a positive real number, there is a Borel probability measure* μ *on* $[0,1]$ *for which*

$$\ln\left(S_\mu^q(2^{-n})\right) = (1-q)\ln(2)\sum_{j=1}^n a_n.$$

Moreover, the net

$$\epsilon \mapsto \frac{1}{q-1}\frac{\ln\left(S_\mu^q(\epsilon)\right)}{\ln(\epsilon)}$$

has the same accumulation points as the sequence

$$n \mapsto \frac{1}{n}\sum_{j=1}^n a_j.$$

Proof. First let's define F, the cumulative distribution function. We start with

$$F(0) = 0 \quad \text{and} \quad F(1) = 1.$$

Choose any ω_n in $\left[0,\frac{1}{2}\right]$ so that

$$\ln\left(\omega_n^q + (1-\omega_n)^q\right) = (1-q)\ln(2)a_n.$$

Define F on the dyadic rationals between 0 and 1 by

$$F\left(\frac{2k+1}{2^{n+1}}\right) = \omega_n F\left(\frac{k}{2^n}\right) + (1-\omega_n)F\left(\frac{k+1}{2^n}\right).$$

Thus F is nondecreasing on the dyadic rationals; set it to 0 on dyadics less than 0 and to one on dyadics greater than 1. For any n and any k with $0 < k < 2^{-n}$ we have

$$\begin{aligned} & F\left(\frac{k}{2^n}\right) - F\left(\frac{k}{2^n} - \frac{1}{2^{n+1}}\right) \\ = \ & F\left(\frac{k}{2^n}\right) - \left(\omega_n F\left(\frac{k-1}{2^n}\right) + (1-\omega_n)F\left(\frac{k}{2^n}\right)\right) \\ = \ & \omega_n\left(F\left(\frac{k}{2^n}\right) - F\left(\frac{k}{2^n} - \frac{1}{2^n}\right)\right). \end{aligned}$$

Since we choose ω_n to the left of $\frac{1}{2}$, we have

$$F\left(\frac{k}{2^n}\right) - F\left(\frac{k}{2^n} - \frac{1}{2^{n+1}}\right) \le \frac{1}{2}\left(F\left(\frac{k}{2^n}\right) - F\left(\frac{k}{2^n} - \frac{1}{2^n}\right)\right).$$

Therefore, if $m \geq 1$,

$$F\left(\frac{k}{2^n}\right) - F\left(\frac{k}{2^n} - \frac{1}{2^{n+m}}\right) \leq \left(\frac{1}{2}\right)^m \left(F\left(\frac{k}{2^n}\right) - F\left(\frac{k}{2^n} - \frac{1}{2^n}\right)\right).$$

Since F is bounded between 0 and 1, we have for any dyadic rational r,

$$\frac{k}{2^n} - \frac{1}{2^{n+m}} \leq r < \frac{k}{2^n} \implies F\left(\frac{k}{2^n}\right) - F\left(r\right) \leq \left(\frac{1}{2}\right)^m.$$

Since F is nondecreasing, this says

$$F(s) = \sup_{s > r \in \mathbb{Z}\left[\frac{1}{2}\right]} F(r)$$

for all s in $\mathbb{Z}[\frac{1}{2}]$.

Let us extend F to $\mathbb{R}$ by

$$F(x) = \sup_{x > r \in \mathbb{Z}\left[\frac{1}{2}\right]} F(r).$$

It is routine to verify that F is left continuous and nondecreasing.

Since F is nondecreasing and left continuous, we have an associated measure μ which satisfies

$$\mu([a,b)) = F(b) - F(a).$$

By design,

$$\begin{aligned} & \mu\left(\left[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}}\right)\right) \\ = \; & F\left(\frac{2k+1}{2^{n+1}}\right) - F\left(\frac{2k}{2^{n+1}}\right) \\ = \; & \omega_n F\left(\frac{k}{2^n}\right) + (1-\omega_n) F\left(\frac{k+1}{2^n}\right) - F\left(\frac{k}{2^n}\right) \\ = \; & (1-\omega_n)\mu\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) \end{aligned}$$

and

$$\begin{aligned} & \mu\left(\left[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}}\right)\right) \\ = \; & F\left(\frac{2k+2}{2^{n+1}}\right) - F\left(\frac{2k+1}{2^{n+1}}\right) \\ = \; & F\left(\frac{k+1}{2^n}\right) - \left(\omega_n F\left(\frac{k}{2^n}\right) + (1-\omega_n) F\left(\frac{k+1}{2^n}\right)\right) \\ = \; & \omega_n \mu\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right). \end{aligned}$$

This tells us

$$S_\mu^q(2^{-n-1}) = (\omega_n^q + (1-\omega_n)^q) S_\mu^q(2^{-n})$$

and so also

$$\ln\left(S_\mu^q(2^{-n-1})\right) = (1-q)\ln(2) a_n + \ln\left(S_\mu^q(2^{-n})\right).$$

Since

$$S_\mu^q(1) = \|\mu\|_1 = 1,$$

induction gives us

$$\ln\left(S_\mu^q(2^{-n})\right) = (1-q)\ln(2)\sum_{j=1}^{n} a_j.$$

With μ as constructed from a_n and q as indicated, Theorem 4.2 Theorem applies to give the final statement in the lemma. □

Theorem 5.2. *Suppose* $0 < q < 1$ *or* $1 < q < \infty$, *and* $\epsilon_n \searrow 0$ *with*

$$\liminf_{n\to\infty} \frac{\ln(\epsilon_{n+1})}{\ln(\epsilon_n)} > 1.$$

Then there is a Borel probability measure on $[0,1]$ *so that*

$$\lim_{n\to\infty} \frac{\ln\left(S_\mu^q(\epsilon_n)\right)}{\ln(\epsilon_n)}$$

exists but

$$\lim_{\epsilon\to 0} \frac{\ln\left(S_\mu^q(\epsilon)\right)}{\ln(\epsilon)}$$

does not.

Proof. By Lemma 4.1, it suffices to prove this in the special case where

$$\epsilon_n = 4^{-k_n}$$

for some $k_n \in \mathbb{N}$. The hypothesis on the ϵ_n translates to the assumption that k_n is nondecreasing, with limit ∞, and that

$$\liminf_{n\to\infty} \frac{k_{n+1}}{k_n} > 1.$$

Select a subsequence k_{n_l} and some $R > 1$ so that

$$k_{n_l+1} \geq R k_{n_l}$$

for all l. Define $a_j = \frac{1}{2}$ for all j except

$$2k_{n_l} < j \leq (k_{n_l} + k_{n_l+1}) \implies a_j = 0$$

and

$$(k_{n_l} + k_{n_l+1}) < j \leq 2k_{n_l+1} \implies a_j = 1$$

With μ as defined above,

$$\frac{1}{q-1}\frac{\ln\left(S_\mu^p(\epsilon_n)\right)}{\ln(\epsilon_n)} = \frac{1}{2k_n}\sum_{j=1}^{2k_n} a_j = \frac{1}{2}.$$

If

$$\eta_n = 2^{-(k_{n_l}+k_{n_l+1})}$$

then

$$\begin{aligned}\frac{1}{q-1}\frac{\ln\left(S_\mu^q(\eta_n)\right)}{\ln(\eta_n)} &= \frac{2k_{n_l}}{k_{n_l}+k_{n_l+1}}\frac{1}{2} + \frac{k_{n_l+1}-k_{n_l}}{k_{n_l}+k_{n_l+1}}0 \\ &= \frac{1}{1+\frac{k_{n_l+1}}{k_{n_l}}} \\ &\geq \frac{1}{1+R}.\end{aligned}$$

Thus

$$\begin{aligned}\liminf_{\epsilon\to 0}\frac{\ln\left(S_\mu^q(\epsilon)\right)}{\ln(\epsilon)} &\leq \liminf_{n\to\infty}\frac{\ln\left(S_\mu^q(\epsilon_n)\right)}{\ln(\epsilon_n)}\\ &= \frac{1}{2}\end{aligned}$$

an

$$\begin{aligned}\limsup_{\epsilon\to 0}\frac{\ln\left(S_\mu^q(\epsilon)\right)}{\ln(\epsilon)} &\geq \limsup_{n\to\infty}\frac{\ln\left(S_\mu^q(\eta_n)\right)}{\ln(\eta_n)}\\ &\geq \frac{1}{1+R}\\ &> \frac{1}{2}.\end{aligned}$$

□

6. Rényi dimension of convolutions

Barbaroux, Germinet and Tcheremchantsev ([1]) establish bounds that relate $D_q^{\pm}(\mu*\nu)$ with $D_q^{\pm}(\mu)$ and $D_q^{\pm}(\nu)$, when q is positive and $q\neq 1$. Here we establish related bounds.

Theorem 6.1. *Suppose μ and ν are Borel measures on $\mathbb{R}^d$. If $1<q<\infty$ and $\frac{1}{q}+1=\frac{1}{r}+\frac{1}{s}$ for some $1<r,s<\infty$, then*

$$D_q^-(\mu*\nu)\geq\frac{q(r-1)}{r(q-1)}D_r^-(\mu)+\frac{q(s-1)}{s(q-1)}D_s^-(\nu).$$

If $0<q<1$, and μ is q=finite, and $\frac{1}{q}+1=\frac{1}{r}+\frac{1}{s}$ for some $0<r,s<1$, then

$$D_q^+(\mu*\nu)\leq\frac{q(r-1)}{r(q-1)}D_r^+(\mu)+\frac{q(s-1)}{s(q-1)}D_s^+(\nu).$$

Proof. We let g again be the standard Gaussian.

Assume $1<r,s<q$. By Young's inequality (Theorem 1.2.12 in [5]),

$$\begin{aligned}&d+\frac{q}{q-1}\frac{\ln(\|g_{\sqrt{2}\epsilon}*(\mu*\nu)\|_p)}{\ln(\sqrt{2}\epsilon)-\ln(\sqrt{2})}\\ &= d+\frac{q}{q-1}\frac{\ln(\|(g_\epsilon*\mu)*(g_\epsilon*\nu)\|_p)}{\ln(\epsilon)}\\ &\geq d+\frac{q}{q-1}\left(\frac{\ln(\|g_\epsilon*\mu\|_r)}{\ln(\epsilon)}+\frac{\ln(\|g_\epsilon*\nu\|_s)}{\ln(\epsilon)}\right) \qquad (7)\\ &= \frac{q(r-1)}{r(q-1)}\left(d+\frac{r}{r-1}\frac{\ln(\|g_\epsilon*\mu\|_r)}{\ln(\epsilon)}\right).\\ &\quad+\frac{q(s-1)}{s(q-1)}\left(d+\frac{s}{s-1}\frac{\ln(\|g_\epsilon*\nu\|_s)}{\ln(\epsilon)}\right)\end{aligned}$$

Notice that

$$0<\frac{q(r-1)}{r(q-1)},\frac{q(s-1)}{s(q-1)}$$

and

$$\frac{q(r-1)}{r(q-1)}+\frac{q(s-1)}{s(q-1)}=1.$$

Now take lim inf of both sides of (7) and apply Guérin's formula (1).

For $0 < q < 1$, the inequality switches in (7), we apply lim sup to both sides, and then use Lemma 2.3. □

7. Best fit slopes

Rather than tracking long-term slopes

$$\frac{\ln\left(S_\mu^q(e^x)\right) - \ln\left(S_\mu^q(e^{x_0})\right)}{x - x_0}$$

to determine a "dimension" for μ, one could look at slope information of the function

$$x \mapsto \ln\left(S_\mu^q(e^x)\right).$$

in many ways. Here we consider the slopes of least-squares best fit lines.

This is not to advocate for or against using least-squares best fit lines for calculating fractal dimensions in practice. See [12, 11] for a discussion. See also [13].

Remark 7.1. Given a measurable function

$$\rho : [0, \infty) \to \mathbb{R}$$

that is bounded on bounded intervals, the slope of closest line over $[0, x]$ to ρ is

$$\frac{6}{x^3}\int_0^x (2t - x)\rho(t)\,dt.$$

Here closest means with respect to the L^2 norm using Lebesgue measure on $[0, x]$.

Remark 7.2. Given a sequence

$$\rho : \mathbb{N} \to \mathbb{R},$$

and any positive real λ, the slope of the least-squares best fit line to

$$\{(0, \rho_0), (\lambda, \rho_1), \ldots, ((n-1)\lambda, \rho_{n-1})\}$$

is

$$\frac{6}{(n^3 - n)\lambda}\sum_{k=0}^{n-1}(2k + 1 - n)\rho_k$$

The following lemmas are needed to work for the partition function, but only depend on bounds from Lemma 3.4.

Definition 7.3. A function $\rho : [0, \infty) \to \mathbb{R}$ is *nearly Lipschitz* if ρ is measurable and there are finite constants A and B so that

$$|\rho(x) - \rho(y)| \le A + B|x - y|.$$

Lemma 7.4. *If $\lambda > 0$ and $\rho : [0, \infty) \to \mathbb{R}$ is nearly Lipschitz, then the sequence*

$$n \mapsto \frac{6}{(n\lambda)^3}\int_0^{n\lambda} (2t - n\lambda)\rho(t)\,dt$$

has the same accumulation points as the net

$$x \mapsto \frac{6}{x^3}\int_0^x (2t - x)\rho(t)\,dt.$$

Moreover, there is a constant C so that

$$1 \leq x \leq y \leq x + \lambda$$

implies

$$\left| \frac{6}{x^3} \int_0^x (2t - x)\rho(t)\, dt - \frac{6}{y^3} \int_0^y (2t - y)\rho(t)\, dt \right| \leq \frac{C}{x}.$$

Proof. Let $A_0 = A + |\rho(0)|$, so that

$$|\rho(t)| \leq A_0 + Bt.$$

A change of variable shows

$$\frac{6}{x^3} \int_0^x (2t - x)\rho(t)\, dt = 6 \int_0^1 (2t - 1)\frac{1}{x}\rho(xt)\, dt.$$

For $x > 0$ and $y > 0$,

$$\begin{aligned} & \left| \int_0^1 (2t-1)\frac{1}{x}\rho(xt)\, dt - \int_0^1 (2t-1)\frac{1}{y}\rho(yt)\, dt \right| \\ = & \left| \int_0^1 (2t-1)\frac{1}{x}\left(\rho(xt) - \rho(yt)\right)\, dt + \int_0^1 \left(\frac{1}{x} - \frac{1}{y}\right)(2t-1)\rho(yt)\, dt \right| \\ \leq & \frac{1}{x}\int_0^1 |2t-1||\rho(xt) - \rho(yt)|\, dt + \frac{|x-y|}{xy}\int_0^1 |2t-1||\rho(yt)|\, dt. \end{aligned}$$

We estimate the first term via

$$\begin{aligned} \int_0^1 |2t-1||\rho(xt) - \rho(yt)|\, dt &\leq \int_0^1 |\rho(xt) - \rho(yt)|\, dt \\ &\leq \int_0^1 A + B|x-y|t\, dt \\ &= A + \frac{1}{2}B|x-y|. \end{aligned}$$

For the second,

$$\begin{aligned} \int_0^1 |2t-1||\rho(yt)|\, dt &\leq \int_0^1 |\rho(yt)|\, dt \\ &\leq \int_0^1 A_0 + Byt\, dt \end{aligned}$$

so

$$\int_0^1 |2t-1||\rho(yt)|\, dt \leq A_0 + \frac{1}{2}By. \tag{8}$$

Therefore

$$\begin{aligned} & \left| \int_0^1 (2t-1)\frac{1}{x}\rho(xt)\, dt - \int_0^1 (2t-1)\frac{1}{y}\rho(yt)\, dt \right| \\ \leq & \frac{1}{x}\left(A + \frac{1}{2}B|x-y|\right) + \frac{|x-y|}{xy}\left(A_0 + \frac{1}{2}By\right). \end{aligned}$$

If we assume

$$x \leq y \leq x + \lambda$$

then

$$\begin{aligned}
& \left|\int_0^1 (2t-1)\frac{1}{x}\rho(xt)\,dt - \int_0^1 (2t-1)\frac{1}{y}\rho(yt)\,dt\right| \\
\leq \;& \frac{1}{x}\left(A+\frac{1}{2}B\lambda\right) + \frac{\lambda}{xy}\left(A_0+\frac{1}{2}By\right) \\
\leq \;& \frac{1}{x}\left(A+\frac{1}{2}B\lambda\right) + \frac{\lambda}{x^2}A_0 + \frac{1}{2}\frac{\lambda}{x}B \\
= \;& (A+B\lambda)\,\frac{1}{x} + (\lambda A_0)\,\frac{1}{x^2}
\end{aligned}$$

□

Lemma 7.5. *If $\lambda > 0$ and $\rho : [0,\infty) \to \mathbb{R}$ is nearly Lipschitz, then the sequence*

$$n \mapsto \frac{6}{(n^3-n)\lambda}\sum_{k=0}^{n-1}(2k+1-n)\rho(\lambda k)$$

has the same accumulation points as the sequence

$$n \mapsto \frac{6}{(n\lambda)^3}\int_0^{n\lambda}(2t-n\lambda)\rho(t)\,dt.$$

Moreover, there is a constant C so that

$$2 \leq n$$

implies

$$\left|\frac{6}{(n\lambda)^3}\int_0^{n\lambda}(2t-n\lambda)\rho(t)\,dt - \frac{6}{(n^3-n)\lambda}\sum_{k=0}^{n-1}(2k+1-n)\rho(\lambda k)\right| \leq \frac{C}{n}.$$

Proof. Let $A_0 = A + |\rho(0)|$, so that

$$|\rho(t)| \leq A_0 + Bt.$$

Equation (8), with $y = n\lambda$, gives us the constant bound

$$\begin{aligned}
\left|\frac{6}{(n\lambda)^3}\int_0^{n\lambda}(2t-n\lambda)\rho(t)\,dt\right| &= \frac{6}{n\lambda}\left|\int_0^1 (2t-1)\rho(n\lambda t)\,dt\right| \\
&\leq \frac{6}{n\lambda}\left(A_0+\frac{1}{2}Bn\lambda\right) \\
&\leq 3\left(\frac{A_0}{\lambda}+B\right)
\end{aligned}$$

as long as $n \geq 2$. Also

$$2 \leq n \implies \left|\frac{n^3}{n^3-n} - 1\right| \leq \frac{2}{n^2}.$$

Therefore, it suffices to estimate the distance from

$$\frac{6}{(n\lambda)^3}\int_0^{n\lambda}(2t-n\lambda)\rho(t)\,dt = \frac{6}{n^3\lambda}\int_0^n (2t-n)\rho(\lambda t)\,dt$$

to

$$\frac{6}{n^3\lambda}\sum_{k=0}^{n-1}(2k+1-n)\rho(\lambda k).$$

For all $n \geq 2$,

$$\begin{aligned}
&\left|\frac{6}{n^3\lambda}\int_0^n (2t-n)\rho(\lambda t)\,dt - \frac{6}{n^3\lambda}\sum_{k=1}^{n-1}(2k+1-n)\rho(\lambda k)\right| \\
= \quad & \frac{6}{n^3\lambda}\left|\sum_{k=0}^{n-1}\int_k^{k+1}(2t-n)\rho(\lambda t)\,dt - \sum_{k=0}^{n-1}\int_k^{k+1}(2k+1-n)\rho(\lambda k)\,dt\right| \\
\leq \quad & \frac{6}{n^3\lambda}\sum_{k=0}^{n-1}\int_k^{k+1}|(2t-n)-(2k+1-n)||\rho(\lambda t)|\,dt \\
& \quad +\frac{6}{n^3\lambda}\sum_{k=0}^{n-1}\int_k^{k+1}|2k+1-n||\rho(\lambda t)-\rho(\lambda k)|\,dt \\
\leq \quad & \frac{6}{n^3\lambda}\sum_{k=0}^{n-1}\int_k^{k+1}|\rho(\lambda t)|+n|\rho(\lambda t)-\rho(\lambda k)|\,dt \\
\leq \quad & \frac{6}{n^3\lambda}\sum_{k=0}^{n-1}\int_k^{k+1}A_0+B\lambda t+n(A+B\lambda(t-k))\,dt \\
= \quad & \left(\frac{6A}{\lambda}+\frac{6|\rho(0)|}{\lambda}\right)\frac{1}{n^2}+\left(9B+\frac{6A}{\lambda}\right)\frac{1}{n}.
\end{aligned}$$

□

Lemma 7.6. *If $\lambda > 0$ and $\rho : [0,\infty) \to \mathbb{R}$ is nearly Lipschitz, then the sequence*

$$n \mapsto \frac{6}{(n^3-n)\lambda}\sum_{k=0}^{n-1}(2k+1-n)\rho(\lambda k)$$

has the same accumulation points as the net

$$x \mapsto \frac{6}{x^3}\int_0^x (2t-x)\rho(t)\,dt.$$

Moreover, there are constants C and D so that

$$D \leq n\lambda \leq x \leq (n+1)\lambda$$

implies

$$\left|\frac{6}{x^3}\int_0^x (2t-x)\rho(t)\,dt - \frac{6}{(n^3-n)\lambda}\sum_{k=0}^{n-1}(2k+1-n)\rho(\lambda k)\right| \leq \frac{C}{n}.$$

Proof. Let C_1 and C_2 be the constants from the last two lemmas. Suppose

$$\max(1, 2\lambda) \leq n\lambda \leq x \leq (n+1)\lambda.$$

Let $y = n\lambda$. We have

$$2 \leq n$$

and

$$0 < y \leq x \leq y + \lambda.$$

Therefore

$$
\begin{aligned}
& \left| \frac{6}{x^3} \int_0^x (2t-x)\rho(t)\,dt - \frac{6}{(n^3-n)\lambda} \sum_{k=0}^{n-1} (2k+1-n)\rho(\lambda k) \right| \\
\leq & \left| \frac{6}{x^3} \int_0^x (2t-x)\rho(t)\,dt - \frac{6}{y^3} \int_0^y (2t-y)\rho(t)\,dt \right| \\
& + \left| \frac{6}{n^3\lambda^3} \int_0^{n\lambda} (2t-n\lambda)\rho(t)\,dt - \frac{6}{(n^3-n)\lambda} \sum_{k=0}^{n-1} (2k+1-n)\rho(\lambda k) \right| \\
\leq & \frac{C_1}{y} + \frac{C_2}{n} \\
= & \left(\frac{C_1}{\lambda} + C_2 \right) \frac{1}{n}.
\end{aligned}
$$

□

Theorem 7.7. *Suppose μ is a finite Borel measure on $\mathbb{R}^d$, and $0 < q < 1$ or $1 < q < \infty$. If $0 < q < 1$ then also suppose μ is q-finite. Suppose $v > 1$. Let*

$$S_\mu^q(\epsilon) = \sum_{k\in\mathbb{Z}} \mu(\epsilon k + \epsilon\mathbb{I})^q.$$

For x in $[0,\infty)$, let m_x and b_x be the real numbers so that

$$t \mapsto m_x t + b_x$$

is the linear function that minimizes

$$\int_{-x}^0 \left(\ln\left(S_\mu^p(e^t)\right) - (m_x t + b_x)\right)^2\, dt.$$

(Here dt refers to Lebesgue measure on $[-x,0]$.) For a natural number n, let $\tilde{m}_n$ and $\tilde{b}_n$ the the real numbers so that

$$t \mapsto \tilde{m}_n t + \tilde{b}_n$$

is the the least-squares best fit line to the pairs

$$\left\{ \left(\ln(v^{-k}), \ln\left(S_\mu^q(v^{-k})\right)\right) \middle| \, k = 0, 1, \ldots, n-1 \right\}.$$

There are constants C and D so that

$$D \leq n\ln(v) \leq x \leq (n+1)\ln(v)$$

implies

$$|m_x - \tilde{m}_n| \leq \frac{C}{n}.$$

In particular,

$$\lim_{x\to\infty} {}^{sup}_{inf}\, m_x = \lim_{n\to\infty} {}^{sup}_{inf}\, \tilde{m}_n.$$

Proof. By Theorem 3.4, there are constants A and B so that

$$\left|\ln\left(S_\mu^q(e^{-x})\right) - \ln\left(S_\mu^q(e^{-y})\right)\right| \leq A + B\,|x-y|$$

for all x and y. Since $\ln\left(S_\mu^q(e^{-x})\right)$ is Borel measurable, we see from Lemma 7.6 that there is are constants C and D so that

$$D \leq n\ln(\nu) \leq x \leq (n+1)\ln(\nu)$$

implies

$$\frac{6}{x^3}\int_0^x (2t-n\lambda)\ln\left(S_\mu^p\left(e^{-t}\right)\right)\,dt$$

is within $\frac{C}{n}$ of

$$\frac{6}{(n^3-n)\ln(\nu)}\sum_{k=0}^{n-1}(2k+1-n)\ln\left(S_\mu^p\left(e^{-k\ln(\nu)}\right)\right).$$

That is,

$$\frac{6}{x^3}\int_0^x (2t-n\lambda)\ln\left(S_\mu^p\left(e^{-t}\right)\right)\,dt \tag{9}$$

is within $\frac{C}{n}$ of

$$\frac{6}{(n^3-n)\ln(\nu)}\sum_{k=0}^{n-1}(2k+1-n)\ln\left(S_\mu^p\left(\nu^{-k}\right)\right). \tag{10}$$

The quantity in (9) gives the slope of the best fit over $[0,x]$ of

$$t\mapsto \ln\left(S_\mu^q(e^{-t})\right),$$

and so

$$m_x = \frac{-6}{x^3\ln(\nu)}\int_0^x (2t-x)\ln\left(S_\mu^q(e^{-t})\right)\,dt. \tag{11}$$

The quantity in (10) gives the slope of the best fit to

$$\left\{\left(k\ln(v),\ln\left(S_\mu^q(v^{-k})\right)\right)\middle|\, k=0,1,\ldots,n-1\right\}$$

and so

$$\tilde{m}_n = \frac{-6}{(n^3-n)\ln(\nu)}\sum_{k=0}^{n-1}(2k+1-n)\ln\left(S_\mu^q(\nu^{-k})\right). \tag{12}$$

We are done. $\square$

Remark 7.8. It is interesting to note some alternative formulas:

$$\tilde{m}_n = \frac{\sum_{k=0}^{n-1}(2k+1-n)\ln\left(S_\mu^q(\nu^{-k})\right)}{\sum_{k=0}^{n-1}(2k+1-n)\ln\left(\nu^{-k}\right)}; \tag{13}$$

$$\tilde{m}_n = \frac{6}{(n^3-n)\ln(\nu)}\sum_{k=1}^{n-1}k(n-k)\left(\ln\left(S_\mu^q(\nu^{-(k-1)})-\ln\left(S_\mu^q(\nu^{-k})\right)\right)\right); \tag{14}$$

$$\tilde{m}_n = \frac{\sum_{k=1}^{n-1}k(n-k)\left(\ln\left(S_\mu^q(\nu^{-k})\right)-\ln\left(S_\mu^q(\nu^{-(k-1)})\right)\right)}{\sum_{k=1}^{n-1}k(n-k)\ln\left(\nu^{-1}\right)}; \tag{15}$$

$$m_x = \frac{\int_0^x (2t-(x))\ln\left(S_\mu^q(\nu^{-t})\right)\,dt}{\int_0^x (2t-x)\ln\left(\nu^{-t}\right)\,dt}. \tag{16}$$

8. More examples

The slope of the least-squares best fit linear approximation to the partition function cannot always be used to determine the Renyi dimension. We show this by the following example. This example is far from what we hope to see in applications.

Lemma 8.1. *For any $1 < q < \infty$, there is a finite Borel measure μ on $[0,1]$ for which*

$$\limsup_{x\to-\infty} \frac{\ln\left(S_\mu^q(e^x)\right)}{x} < \limsup_{x\to-\infty} m_x,$$

where

$$m_x t + b_x \approx \ln\left(S_\mu^q(e^t)\right) \quad (-x \le t \le 0)$$

is the least-squares best fit line. More specifically, m_x and b_x minimize

$$\int_{-x}^{0} \left(\ln\left(S_\mu^q(e^t)\right) - (m_x t + b_x)\right)^2 \, dt.$$

For any $0 < q < 1$, there is a finite Borel measure μ on $[0,1]$ for which

$$\liminf_{\epsilon\to 0} \frac{\ln\left(S_\mu^q(e^x)\right)}{x} > \liminf_{x\to-\infty} m_x,$$

Proof. We will use the μ associated with a sequence a_n, as in Lemma 5.1.

Let

$$a_1 = \frac{30}{47}$$

and

$$a_k = \begin{cases} 0 & \text{if } 48^n < k \le 12(48^n), \text{ any } n \in \mathbb{N} \\ 1 & \text{if } 12(48^n) < k \le 36(48^n), \text{ any } n \in \mathbb{N} \\ \frac{1}{2} & \text{if } 36(48^n) < k \le 48(48^n), \text{ any } n \in \mathbb{N} \end{cases}.$$

When $1 < q < \infty$,

$$\limsup_{\epsilon\to 0} \frac{\ln\left(S_\mu^q(\epsilon)\right)}{\ln(\epsilon)} = \limsup_{n\to 0} (q-1)\ln(2)\frac{1}{n}\sum_{k=1}^{n} a_k$$

and

$$\limsup_{\epsilon\to 0} m_x = \limsup_{n\to 0} \frac{6(q-1)\ln(2)}{n^3}\sum_{k=1}^{n-1} k(n-k)a_k.$$

Therefore we really only need to show that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k < \limsup_{n\to\infty} \frac{6}{n^3}\sum_{k=1}^{n-1} k(n-k)a_k.$$

For $q < 1$ the $(q-1)$ reverses the inequalities and turns each $\liminf$ into a $\limsup$, so the desired inequality reduces to the same thing.

For all n,

$$\begin{aligned} \frac{1}{48^{n+1}} \sum_{k=1}^{48^{n+1}} a_k &= \frac{1}{48^{n+1}} \left[\sum_{k=1}^{48^n} a_k + 24(48^n) + 12(48^n)\frac{1}{2}\right] \\ &= \frac{1}{48}\left[\frac{1}{48^n}\sum_{k=1}^{48^n} a_k + 30\right]. \end{aligned}$$

Since

$$\frac{1}{48}\left[\frac{30}{47}+30\right]=\frac{30}{47}$$

we have found

$$\frac{1}{48^n}\sum_{k=1}^{48^n} a_k=\frac{30}{47}$$

for all n. The terms of value 0 will cause the average to fall until index $12(48^n)$. At this point, the average will be

$$\frac{1}{12}\frac{30}{47}+\frac{11}{12}0=\frac{5}{94}$$

The next $12(48^n)$ terms are of value 1, so the average rises to

$$\frac{1}{3}\frac{5}{94}+\frac{2}{3}1=\frac{193}{282}.$$

Next the average falls, due the terms of value $\frac{1}{2}$, until it is back to $\frac{30}{47}$. Therefore,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} a_k=\frac{193}{282}.$$

We now need just a decent estimate on

$$\limsup_{n\to\infty}\frac{6(p-1)}{n^3}\sum_{k=1}^{n-1} k(n-k)a_k.$$

Indeed, since $a_n>0$ for all n, we find

$$\begin{aligned}
& \limsup_{n\to\infty}\frac{6}{n^3}\sum_{k=1}^{n-1} k(n-k)a_k \\
\geq\ & \limsup_{n\to\infty}\frac{6}{(48^n)^3}\sum_{k=1}^{48^n} k(48^n-k)a_k \\
>\ & \limsup_{n\to\infty} 6\frac{\sum_{k=12(48^{n-1})+1}^{36(48^{n-1})} k(48^n-k)+\sum_{k=36(48^{n-1})+1}^{48^n} k(48^n-k)\frac{1}{2}}{(48^n)^3} \\
=\ & \lim_{n\to\infty} 6\sum_{k=12(48^{n-1})+1}^{36(48^{n-1})}\frac{k}{48^n}\left(1-\frac{k}{48^n}\right)\frac{1}{48^n} \\
& \quad +\lim_{n\to\infty} 3\sum_{k=36(48^{n-1})+1}^{48^n}\frac{k}{48^n}\left(1-\frac{k}{48^n}\right)\frac{1}{48^n} \\
=\ & 6\int_{\frac{1}{4}}^{\frac{3}{4}} t\,(1-t)\ dt+3\int_{\frac{3}{4}}^{1} t\,(1-t)\ dt \\
=\ & 6\left(\frac{1}{2}\left(\frac{3}{4}\right)^2-\frac{1}{3}\left(\frac{3}{4}\right)^3\right)-3\left(\frac{1}{2}\left(\frac{1}{4}\right)^2-\frac{1}{3}\left(\frac{1}{4}\right)^3\right) \\
=\ & \frac{49}{64}.
\end{aligned}$$

Thus
$$\limsup_{\epsilon\to 0} \frac{\ln\left(S_\mu^q(\epsilon)\right)}{\ln(\epsilon)} = \frac{193}{282}(q-1) = \frac{6176}{9024}(q-1)$$
and
$$\limsup_{\epsilon\to 0} m_x > \frac{49}{64}(q-1) = \frac{6909}{9024}(q-1).$$
□

9. Modified Rényi Dimensions

The theory of regular variation and its extensions ([2]) give many ways to measure how closely a function f behaves like various powers x^c near ∞. Regular variation forces f to behave like a single power x^δ.

More realistic classes are those of extended variation and O-regularly varying functions. Both classes allow f to behave like x^c for c in a range (α, β), but they differ on the meaning of "behave." (This is a bit vague. See[2].) The extended real numbers α and β are called Karamata indices in the case where f is of extended variation. For the class of O-regularly varying functions, these are called the Matuszewska indices.

Guido and Isola ([7, 8, 9]) have used the Matuszewska indices to define a new local fractal dimension. Stern ([16]) has suggested generally that the theories of extended variation and O-regular variation be applied to global fractal dimensions.

The example in Section 8 is rather unnatural. It can perhaps be explained away if we use Matuszewska indices to describe the "slope at infinity" of the partition function.

We use the following as a working definition of the Matuszewska indices. It is equivalent to the standard definition, c.f. pages 68–73 of [2].

Definition 9.1. Suppose
$$f : [0,\infty) \to (0,\infty)$$
is any function. The *upper Matuszewska index of f is*
$$\alpha(f) = \left\{ \alpha \in \mathbb{R} \,\middle|\, \exists X, C \text{ s.t. } y \geq xgeqX \implies f(y) \leq f(x) C \left(\frac{y}{x}\right)^\alpha \right\}.$$
Here X are C are to be understood to be positive real numbers. The *lower Matuszewska index of f is*
$$\beta(f) = \left\{ \beta \in \mathbb{R} \,\middle|\, \exists X, C \text{ s.t. } y \geq x \geq X \implies f(y) \geq f(x) C \left(\frac{y}{x}\right)^\beta \right\}.$$

It is easy to show that
$$\beta(f) \leq \liminf_{x\to\infty} \frac{\ln(f(x))}{\ln(x)} \leq \limsup_{x\to\infty} \frac{\ln(f(x))}{\ln(x)} \leq \alpha(f).$$
Again, see [2]. The middle numbers are the so-called order of f. In an unfortunate clash of terminology, the "upper and lower Rényi dimensions of order q" are the upper and lower orders of
$$x \mapsto \left(S_\mu^q(x^{-1})\right)^{\frac{1}{1-q}}.$$
Perhaps it is better to refer to q as the index.

Definition 9.2. If μ is a finite measure, and if $0 < q < 1$ or $1 < q < \infty$, the *upper* and *lower Matuszewska Dimensions of index* q are the upper and lower Matuszewska indices of the function

$$x \mapsto \left(S_\mu^q(x^{-1})\right)^{\frac{1}{1-q}},$$

denoted $D_q^{++}(\mu)$ and $D_q^{--}(\mu)$ respectively.

Theorem 9.3. *Suppose μ is a finite Borel measure on $\mathbb{R}^d$. If $1 < q < \infty$, or if $0 < q < 1$ and μ is q-finite, then the partition function $S_\mu^q(x^{-1})$ is of extended variation and*

$$0 \le D_q^{--}(\mu) \le D_q^{-}(\mu) \le D_q^{+}(\mu) \le D_q^{++}(\mu) \le d.$$

Proof. The second and fourth inequalities come from the general facts about order and Matuszewska indices. The middle is even more standard. The outer inequalities are really just restatements of those in Theorem 3.4.

Equivalently, these inequalities show that the upper and lower Matuszewska indices of $S_\mu^q(x^{-1})$ are bounded between and 0 and $1 - q$. Since $S_\mu^q(x^{-1})$ is measurable, we can apply [2, Theorem 2.1.7] to conclude that $S_\mu^q(x^{-1})$ is of extended variation. □

Remark 9.4. In the example of Section 8:

$$D_q^{--}(\mu) = 0,$$

$$D_q^{-}(\mu) = \frac{5}{94},$$

$$D_q^{+}(\mu) = \frac{193}{282},$$

$$D_q^{++}(\mu) = 1.$$

Thus the upper and lower Matuszewska dimensions dismiss μ from "being fractal" more resoundingly than do the upper and lower Rényi dimensions.

References

[1] J.-M. Barbaroux, F. Germinet, and S. Tcheremchantsev, *Generalized fractal dimensions: equivalences and basic properties*, J. Math. Pures Appl. (9) **80** (2001), no. 10, 977–1012, MR1876760 (2002i:28009), Zbl 1050.28006.

[2] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and its Applications, vol. 27, Cambridge University Press, Cambridge, 1987, MR898871 (88i:26004), Zbl 0617.26001.

[3] P. J. Burt and E. H. Adelson, *The Laplacian Pyramid as a compact image code*, IEEE Transactions on Communications **COM-31,4** (1983), 532–540.

[4] G. B. Folland, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995, MR1397028 (98c:43001), Zbl 0857.43001.

[5] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson/Prentice, 2004.

[6] Charles-Antoine Guérin, *A note on the generalized fractal dimensions of a probability measure*, J. Math. Phys. **42** (2001), no. 12, 5871–5875, MR1866693 (2002h:28010), Zbl 1008.28006.

[7] D. Guido and T. Isola, *Dimensions and singular traces for spectral triples, with applications to fractals*, J. Funct. Anal. **203** (2003), no. 2, 362–400, MR2003353 (2005b:58038), Zbl 1031.46081.

[8] ———, *Tangential dimensions. I. Metric spaces*, Houston J. Math. **31** (2005), no. 4, 1023–1045 (electronic), MR2175420.

[9] ———, *Tangential dimensions. II. Measures*, Houston J. Math. **32** (2006), no. 2, 423–444 (electronic).

[10] H. G. E. Hentschel and Itamar Procaccia, *The infinite number of generalized dimensions of fractals and strange attractors*, Phys. D **8** (1983), no. 3, 435–444, MR719636 (85a:58064), Zbl 0538.58026.

[11] N. C. Kenkel and D. J. Walker, *umanitoba.ca/ faculties/ science/ botany/ labs/ ecology/ fractals/ fractal.html*.

[12] ———, *Fractals in the biological sciences*, Coenoses **11** (1996), 77–100.

[13] Benoit B. Mandelbrot, *Measures of fractal lacunarity: Minkowski content and alternatives*, Fractal geometry and stochastics (Finsterbergen, 1994), Progr. Probab., vol. 37, Birkhäuser, Basel, 1995, pp. 15–42, MR1391969 (97d:28009), Zbl 0841.28010.

[14] R. Riedi, *An Improved Multifractal Formalism and Self-affine Measures*, Ph.D. thesis, ETH Zurich, 1993.

[15] Rolf Riedi, *An improved multifractal formalism and self-similar measures*, J. Math. Anal. Appl. **189** (1995), no. 2, 462–490, MR0315747 (47 #4296).

[16] I. Stern, *On Fractal Modeling in Astrophysics: The Effect of Lacunarity on the Convergence of Algorithms for Scaling Exponents*, Astronomical Data Analysis Software and Systems VI, ASP Conference Series, vol. 125, Astronomical Society of the Pacific, Basel, 1997, pp. 222–226.

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